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The decay of slow viscous flow

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Abstract. A weak force acts within a viscous fluid for a finite time leading to a slow flow with negligible viscous effects; the force then ceases so that the fluid returns steadily to a state of rest under the action of diffusion. In the model developed, the force is equivalent in time to a delta function mathematically, having the form of a pulse physically; singular solutions such as rotlets and stokeslets are introduced to simplify the calculations and their use can be justified as representing solid bodies. Here we solve the transient Stokes flow equations to find the behaviour in a number of different situations, the rates of decay are computed, and the nature of the final motion described. A number of general conclusions are deduced from these examples.

1. Introduction

The purpose of the present paper is to consider the nature of the decay in time of a slow viscous (Stokes) flow. The situation we are interested in specifically concerns a volume of fluid which has been subject to a force for a finite period of time, thereby developing a motion in the fluid; this force then ceases to act so that the motion comes to rest freely – rather than through the possibility that the force itself tends to zero steadily, so controlling the decay. We assume that the velocities developed during the time of action are sufficiently small so that the flow can be taken to be slow in the sense that the non-linear inertial terms in the Navier–Stokes equations are negligible throughout. Nevertheless, there will be a wide range of conditions under which a fluid passes through its final stages before coming to rest in a manner governed by such a linear analysis; the major characteristic is that the flow is dominated by diffusion. The main questions raised concern both the rate at which the motion in the fluid comes to rest, and the nature of the final behaviour present as the time tends to infinity; this will be the behaviour which can be observed at the end, and will lead to the final drag or torque which the fluid exerts on the boundary of the container.

Now the Stokes flows represent motions where the Reynolds number is effectively zero, and such behaviours are invariably considered as being time-independent basically because when there is some variation over time it can be accommodated through taking the flow to be quasi-steady, with the external variable (the velocity of the body, for example) being the time-dependent parameter in an otherwise steady flow. Stokes [1] presented the solution for the flow of a uniform stream past a sphere, and this result was extended by Basset [2] who calculated the effect of a variable velocity on the drag through the so-called 'memory term'; this quantity can certainly be considered negligible under a wide range of conditions.

The intention here is to consider intrinsically unsteady flows through investigating the effect of weak pulses on motions in a viscous fluid, where the term 'pulse' is used to represent the rapid motion of a body for a short period of time. More precisely, when a body is moved with a constant velocity U_0 for a fixed time T_0, we are concerned with the limiting situation where U_0 is large and T_0 is small such that the product U_0T_0 has finite magnitude;

clearly this product is a measure of the distance a which is travelled by the body. When the viscosity of the fluid is ν, then we can define a Reynolds number Re by $\mathrm{Re} = U_0 a \nu^{-1} = U_0^2 T_0 \nu^{-1}$ for such situations.

If we define lengths by ax, ay, az in cartesian co-ordinates, and time by $T_0 t$, then the inertial terms in the Navier–Stokes equations are negligible when this Reynolds number is small. We consider flows which satisfy the resultant equations

$$u_x + v_y + w_z = 0\,, \tag{1.1}$$

$$u_t = -p_x + u_{xx} + u_{yy} + u_{zz} \neq \emptyset\,, \tag{1.2}$$

$$v_t = -p_y + v_{xx} + v_{yy} + v_{zz} \neq \emptyset\,, \tag{1.3}$$

$$w_t = -p_z + w_{xx} + w_{yy} + w_{zz} \neq \emptyset\,, \tag{1.4}$$

where u, v, w are the non-dimensional velocities in the x, y, z directions and p is the pressure, to be the consequences of 'weak pulses'.

This model is taken to represent any force, or forced motion, in the fluid which acts for a finite time and then stops. We find that the decay can be rather rapid in many of the bounded geometries, and so the dominant term for large t in the mathematical solution does represent the observable flow for much of the time. It can also be anticipated that the results have value for pulses of magnitude larger than those which strictly satisfy our requirement of being weak.

The equations (1.1)–(1.4) have been considered in a number of different contexts (see, for example, Kim and Karrila [3]). Certainly the most well-known correspond to Rayleigh flows, where an infinite cylinder with constant cross-section moves parallel to its generators. Diffusion alone acts, and so these equations represent exact solutions to the Navier–Stokes equations. The classical result of Stokes [1] himself describes the effect of impulsively starting an infinite flat plate, and the essential role of the diffusive variable $yt^{-1/2}$ (when the motion is parallel to the x-axis) is emphasised. The later work of Batchelor [4] and Hasimoto [5] catalogued many equivalent solutions for more complicated geometries; much of the mathematics required had been developed earlier for heat conduction problems (cf. Carslaw and Jaeger [6]). When, for example, the infinite plate $y = 0$ is moved parallel to the x-axis then the velocity from a pulse of unit magnitude at $t = 0$ is given by

$$u = \frac{y}{2\pi^{1/2} t^{3/2}} \exp\left(-\frac{y^2}{4t}\right) \tag{1.5}$$

on differentiating the solution of Stokes (see Fig. 1). As $t \to \infty$, with $y = O(1)$, $u \cong \frac{1}{2}\pi^{-1/2} y t^{-3/2}$, and the velocity decays algebraically in time with the final motion being a simple shear with $u \propto y$. All the solutions of Batchelor and Hasimoto can be differentiated similarly to give pulse solutions. In the limiting situation of $U_0 \to \infty$ and $T_0 \to 0$ such that $U_0 T_0 = O(1)$ the force leading to the pulse is represented by the Dirac delta function, and so the solutions for such situations can be calculated from the corresponding solutions for the impulsively started flow, where the force is represented by the Heaviside step function.

In his text, Oseen [7] developed solutions for a large number of linear problems in hydrodynamics using fundamental solutions (Greens functions) as the basic building blocks, with general bodies being represented by a distribution of these singularities. A substantial section was concerned with unsteady flows. More recently, Smith [8, 9] and Hansen [10] have described some particular two-dimensional flows (derived from the equations (1.1)–(1.3))

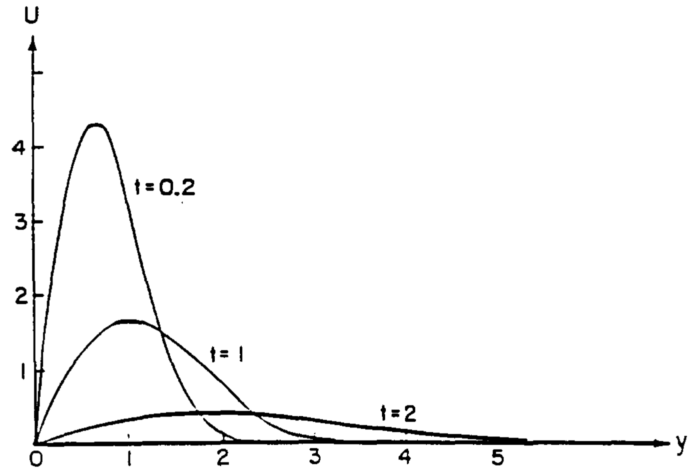

Fig. 1. Velocity profile for the pulse of unit strength as given by (1.5).

corresponding to impulsively started motions; the former were particularly concerned with the development of separated flow.

Here we consider only three-dimensional behaviours; however, the mathematical approach is essentially similar, is sufficiently straightforward, and so the calculations are only sketched for the sake of brevity. We present solutions for a variety of different problems in Sections 2–4, with comments on the physical results; in the final section some general conclusions which can be implied from these examples are presented.

2. Rotational flows

In the cylindrical co-ordinate system (ρ, ϕ, z), with $x = \rho \cos\phi$, $y = \rho \sin\phi$, we write the velocities in radial, axial and azimuthal directions as U, W and V, respectively. We first investigate flows with rotation about the z-axis for small Reynolds numbers where U and W can be set as zero; although the centrifugal force due to the rotation does lead to a flow in the azimuthal plane, these velocities are small as $O(\mathrm{Re})$, and can be neglected.

To begin, when a point rotlet is situated at the origin the governing equation is

$$V_t = V_{\rho\rho} + \rho^{-1}V_\rho - \rho^{-2}V + V_{zz} - \rho^{-1}\delta'(\rho)\delta(z)\delta(t)\,. \tag{2.1}$$

This is the limiting situation of a rotlet of strength Ω for the time $0 \leq t < \tau$, where $\Omega \to \infty$, $\tau \to 0$ such that $\Omega\tau \to 1$. A Laplace transform in t, Fourier cosine transform in z, and Hankel transform in ρ leads to

$$V = \frac{1}{2\pi^2 i}\int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{e}^{st}\,\mathrm{d}s \int_0^\infty \cos\alpha z\,\mathrm{d}\alpha \int_0^\infty \frac{k^2 J_1(k\rho)}{k^2 + \alpha^2 + s}\,\mathrm{d}k\,, \tag{2.2}$$

which can be evaluated exactly for

$$V = \frac{\rho}{8\pi^{1/2}t^{5/2}}\exp\left(-\frac{r^2}{4t}\right), \quad t > 0\,, \tag{2.3}$$

where $\rho = r\sin\theta$, $z = r\cos\theta$. Diffusion of the pulse away from the origin is the sole process at work; as $t \to \infty$ there is solid body rotation at finite distances, with $V \propto \rho$, and an algebraic decay.

One point to check is whether the solution (2.3) does represent the flow due to a rotating body. In the model problem the rotlet acts for $t = 0$, and then is no longer present once $t > 0$; however, with a solid body which has ceased to rotate the no-slip conditions must still be satisfied on the body for all $t > 0$. Therefore, we now investigate the flow when the sphere $r = 1$ is given a unit pulse of angular velocity; that is, we write $V = \chi(r, t)\sin\theta$ with

$$\chi_t = \chi_{rr} + 2r^{-1}\chi_r - 2r^{-2}\chi \tag{2.4}$$

subject to $\chi = \delta(t)$ on $r = 1$, $\chi \to 0$ as $r \to \infty$. When a Laplace transform of (2.4) is taken, it follows that

$$\bar{\chi} = \frac{1 + \sigma r}{(1 + \sigma)r^2}\,e^{-\sigma(r-1)},$$

where here, and throughout the paper, we write $\sigma = s^{1/2}$; this leads to

$$\chi = \frac{r-1}{r^2}\left\{\frac{1}{(\pi t)^{1/2}}\left(\frac{r}{2t} - 1\right)\exp\left(-\frac{(r-1)^2}{4t}\right) + e^{r-1+t}\operatorname{erfc}\left(\frac{r-1}{2t^{1/2}} + t^{1/2}\right)\right\}. \tag{2.5}$$

When the asymptotic approximation is taken of (2.5) for $r \gg 1$, it is confirmed that it does lead to (2.3) as required; a rotlet can be used legitimately to develop an understanding of the flow due to small axisymmetric bodies rotating inside a container. Therefore, we now use (2.3) to find different decay rates when the rotlet pulse is positioned within different geometries.

(a) When the rotlet is situated on the axis $\rho = 0$ and $z = c$ $(0 < c < 1)$, between the rigid plates $z = 0$ and $z = 1$, then we must solve (2.1) subject to the conditions $V = 0$ on $z = 0, 1$. The solution is simplified when we observe that writing

$$V = \frac{\rho}{8t^2}\exp\left(-\frac{\rho^2}{4t}\right)\left[(\pi t)^{-1/2}\exp\left(-\frac{(z-c)^2}{4t}\right) + V_1(z, t)\right] \tag{2.6}$$

leads to the reduced equation $V_{1t} = V_{1zz}$; the first term corresponds to (2.3). After completing the details, we are left with the expression

$$V = \frac{1}{4}\rho t^{-2}\exp\left(-\frac{\rho^2}{4t}\right)\sum_{n=1}^{\infty}\sin(n\pi c)\sin(n\pi z)\,e^{-n^2\pi^2 t}; \tag{2.7}$$

the inverse Laplace transform reduces to the infinite sum of residues at simple poles on the negative real axis in the complex s-plane. For small time t the rotlet solution (2.3) dominates, and (2.7) can in fact be written as (2.3) plus an infinite set of images; however, the representation (2.7) shows how the presence of the plates quickly dampens the pulse until the final decay is complete. For large t the decay is exponential with $V \propto \rho\sin(\pi z)\,e^{-\pi^2 t}$; this represents the solution to $V_{zz} + \lambda^2 V = 0$, with $V = 0$ on $z = 0, 1$, corresponding to the smallest eigenvalue λ.

(b) When the rotlet is situated axisymmetrically at the origin inside the circular cylinder $\rho = 1$, then following the procedure in (a) above we can write

$$V = \frac{1}{8t^{1/2}} \exp\left(-\frac{z^2}{4t}\right) \left[\frac{\rho}{\pi^{1/2} t^2} \exp\left(-\frac{\rho^2}{4t}\right) + V_2(\rho, t)\right], \tag{2.8}$$

where V_2 satisfies $V_{2t} = V_{2\rho\rho} + \rho^{-1} V_{2\rho} - \rho^{-2} V_2$. After the Laplace transform is taken, it is seen that the integral for the inverse is again evaluated as a sum of residues on the negative real s-axis at the zeros of $I_1(\sigma)$, with the result

$$V = \frac{\pi^{3/2}}{2t^{1/2}} \exp\left(-\frac{z^2}{4t}\right) \sum_{n=1}^{\infty} \frac{\gamma_n^2 Y_1(\gamma_n)}{J_0(\gamma_n)} J_1(\gamma_n \rho)\, e^{-\gamma_n^2 t}, \tag{2.9}$$

where γ_n $(n = 1, 2, 3, \ldots)$ are the roots of $J_1(\gamma) = 0$; numerically, $\gamma_1 \cong 3.832$, $\gamma_2 \cong 7.016$, $\gamma_3 \cong 10.173$, etc. The velocity is given as a sum of eigenfunctions $J_1(\gamma_n \rho)$; the larger the value of n the more zeroes there are in $0 \leqslant \rho \leqslant 1$, and so the quicker the damping as t increases, with the final decay showing $V \propto J_1(\gamma_1 \rho)$ throughout the cylinder for finite z.

When the rotlet is situated inside the cylinder more generally at the point $(c, 0, 0)$, with $0 < c < 1$, the same analytical approach can be followed, though now also requiring the stream function to be given as Fourier series in $\cos(m\phi)$ for integers m. The details are extensive, but here it is sufficient to state that the inverse transform is gained through residues to show the velocities U and V (axial symmetry has been lost) as double infinite series over m and n including the terms $J_m(\gamma_{mn}\rho) \sin(m\phi) \exp(-\gamma_{mn}^2 \rho - \frac{1}{4} z^2/t)$ for U (with $\cos(m\phi)$ replacing $\sin(m\phi)$ for V), where γ_{mn} is the nth root of $J_{m+1}(\gamma) = 0$. The final motion, therefore, corresponds to the lowest eigenvalue $\gamma_{01} \cong 3.832$ – this is just that found earlier when the rotlet was positioned on the axis. Consequently, it is sufficient to investigate just the simpler axisymmetric problem when only the ultimate behaviour is required.

(c) When the rotlet is at the centre of the sphere $r = 1$, then the spherical symmetry allows us to write $V = \chi(r, t) \sin \theta$ again, where χ satisfies (2.4). Completing the details leads to

$$\chi = \frac{1}{r^2} \sum_{n=1}^{\infty} (\delta_n^2 + 1) \left\{\cos \delta_n r - \frac{\sin \delta_n r}{\delta_n r}\right\} e^{-\delta_n^2 t}, \tag{2.10}$$

where δ_n are the roots of $\tan \delta = \delta$ for $n = 1, 2, 3, \ldots$; numerically $\delta_1 \cong 4.493$, $\delta_2 \cong 7.725$, $\delta_3 \cong 10.904$, with $\delta_n \cong (n + \frac{1}{2})\pi$.

Because of the degree of symmetry this situation can be generalized to consider the flow between concentric spheres, and so the manner in which the exponential decay becomes algebraic as the outer boundary grows in size can be observed. Fluid is contained between the spheres $r = c, d$ (with $d > c$), and the motion is created by the inner sphere being given a rotation by a unit pulse about the z-axis. That is, we set $\chi = \delta(t)$ on $r = c$ and $\chi = 0$ on $r = d$, which leads to

$$\bar{\chi} = \frac{c^2}{r^2} \frac{(d\sigma^2 r - 1) \sinh \sigma(d - r) + \sigma(d - r) \cosh \sigma(d - r)}{(d\sigma^2 c - 1) \sinh \sigma(d - c) + \sigma(d - c) \cosh \sigma(d - c)}. \tag{2.11}$$

When the inverse transform is taken, the total contribution is derived from the simple poles given by the zeros of

$$(dc\lambda^2 + 1) \tan \lambda(d - c) = \lambda(d - c); \tag{2.12}$$

there is an infinite set of roots λ_n, with the slowest decay proportional to $e^{-\lambda_1^2 t}$ where the

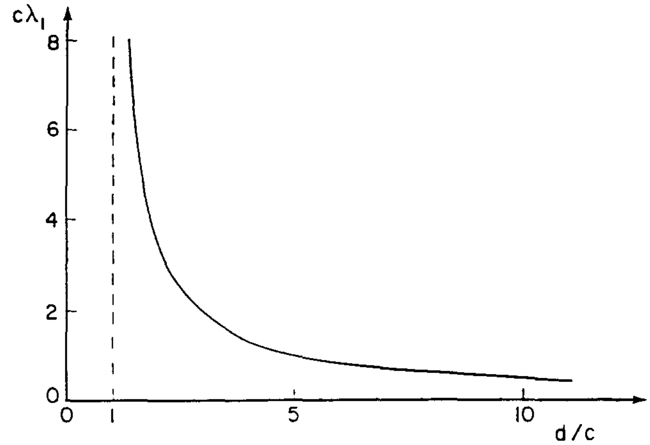

Fig. 2. The smallest root λ_1 of the eigenvalue equation (2.12) for different values of the ratio d/c.

value of λ_1 for different values of the ratio d/c is given by Fig. 2. As $d/c \to \infty$, $\lambda_1 \to 0$ and the exponential decay becomes algebraic; asymptotically, $\lambda_1 \cong 4.493(d-c)^{-1}$ as $d/c \to \infty$.

It can be observed from the three examples with a rotlet how the more confined the domain implies the more rapid the decay as a general principle: from $e^{-9.870t}$ for the flow between two plates a unit distance apart, to $e^{-14.682t}$ for the flow inside a circular cylinder of unit radius, to $e^{-20.191t}$ for the flow inside a sphere of unit radius.

(d) One case we include without axial symmetry concerns a rotlet pulse at the point $(1, 0, 0)$ in cartesian co-ordinates, where the axis of rotation is vertical, with a wall along the plane $x = 0$. Because the motion is completely due to the shear created by the rotlet the axial velocity w is identically zero, and so we can define a stream function $\psi(x, y, z, t)$ by $u = \psi_y$, $v = -\psi_x$ with $\omega = \psi_{xx} + \psi_{yy}$; then the vorticity satisfies

$$\omega_t = \omega_{xx} + \omega_{yy} + \omega_{zz} .$$

When we define

$$\bar{\bar{\bar{\psi}}}(x, \beta, \gamma, s) = \int_0^\infty e^{-st}\, dt \int_0^\infty \cos \gamma z \, dz \int_0^\infty \cos \beta y \psi(x, y, z, t)\, dy ,$$

it is seen that

$$\frac{4}{\pi} \bar{\bar{\bar{\psi}}} = \frac{e^{-\kappa|x-1|}}{\kappa} - \frac{2\, e^{-\kappa-\beta x}}{\kappa - \beta} + \frac{(\kappa+\beta)\, e^{-\kappa(x+1)}}{\kappa(\kappa-\beta)} , \qquad \kappa^2 = \beta^2 + \gamma^2 + s . \tag{2.13}$$

The inverse transform can be completed, and it is found that $\psi \cong \frac{1}{8}\pi^{-1} y^2 t^{-5/2}$ as $t \to \infty$ for finite y; this is a linear shear flow. Of particular interest is the presence of separated flow for, from (2.13), it is found that

$$(\omega)_{x=0} = -\frac{1}{8\pi^{1/2} t^{3/2}} \exp\left(-\frac{y^2+z^2+1}{4t}\right) \left[1 - 2t + 2\left(\frac{t}{\pi}\right)^{1/2} \left\{ e^{\zeta^2} - 2\zeta \int_0^\zeta e^{u^2}\, du \right\}\right] , \tag{2.14}$$

where $\zeta = \frac{1}{2} y t^{-1/2}$. It follows from (2.14) that the flow separates instantaneously from the wall at $y = 0$ and lasts until $t = t_0 \cong 1.089$, with the maximum extent of the separation point

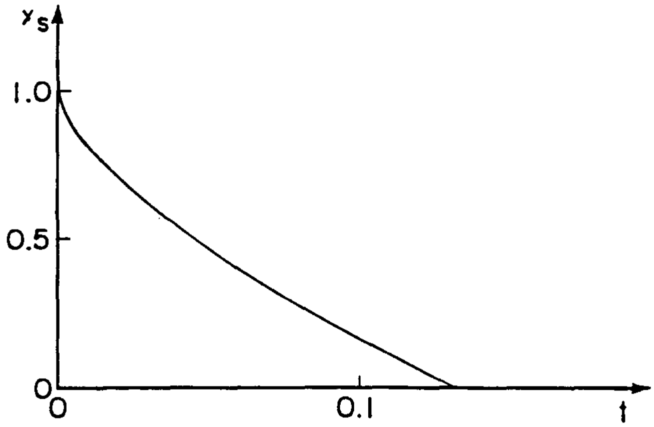

Fig. 3. The position of the separation point $y_s(t)$ for the pulse flow due to a rotlet in front of a plane wall.

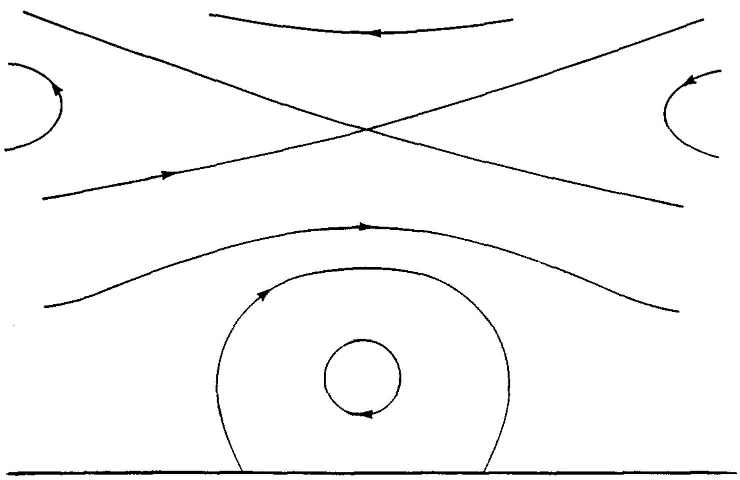

Fig. 4. The streamlines during the separated part of the flow due to a rotlet in front of a plane wall.

from the origin begin $(y_s)_{max} \cong 1.307$; the graph of $y_s(t)$ is given in Fig. 3. The streamlines for the flow during the time $0 < t < t_0$ are found from (2.13) and represented in Fig. 4; the maximum displacement is $(x_s)_{max} \cong 2.162$. The growth and decay of this eddy is a distinctive feature.

Another point to emphasize is that the motion here is essentially two-dimensional with the z-variable present only as the diffused scaling $\exp(-z^2/4t)$; this follows from the observation that ψ in (2.13) is a function of x, β and $\gamma^2 + s$ alone.

3. Axisymmetric streaming flows

Another group of problems we consider concerns axisymmetric motion in the azimuthal plane where the motion is described by the radial and axial velocities U, W, with V being zero. Consequently, the stream function $\Psi(\rho, z, t)$ can be defined by $U = \rho^{-1}\Psi_z$, $W = -\rho^{-1}\Psi_\rho$, and the vorticity

$$\Omega = L_{-1}\Psi \quad \text{where} \quad L_{-1} \equiv \frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}.$$

If there is a stokeslet directed along the z-axis at the origin in an unbounded fluid then the governing equation is

$$\Omega_t = L_{-1}\Omega + \rho^{-1}\delta'(\rho)\delta(z)\delta(t)\,; \tag{3.1}$$

the singular term represents a point source of momentum which is present in the limit as $t \to 0$. A stokeslet pulse attempts to model any small body which moves quickly over a finite distance and then stops. The solution of (3.1) is calculated through taking transforms in t, z and ρ (as described for the rotlet in Section 2), and leads directly to

$$\Psi = \frac{1}{4\pi}\left\{\frac{1}{r}\,\mathrm{erf}\left(\frac{r}{2t^{1/2}}\right) - \frac{1}{(\pi t)^{1/2}}\exp\left(-\frac{r^2}{4t}\right)\right\}\sin^2\theta\,; \tag{3.2}$$

$r^2 = \rho^2 + z^2$. For small time t, the approximation to (3.2) shows $\Psi \cong (4\pi)^{-1}\rho^2 r^{-3}$; the effect of the momentum pulse is to instantaneously create a dipole, and the subsequent motion shows its decay through diffusion. For large t we have $\Psi \cong \frac{1}{24}(\pi t)^{-3/2} r^2 \sin^2\theta$ for $r = O(1)$; the final motion represents a uniform stream which decays algebraically in t, and the error term is proportional to $r^4 \sin^2\theta$ – which indicates parabolic streaming flow.

It is necessary, as before, to confirm the validity of the model of a stokeslet as representing a finite body through calculating the complementary problem of the flow past a sphere. That is, we investigate the limiting situation where a uniform stream with velocity $\tilde{U}$ acts for $0 \le t < \tilde{t}$, with $\tilde{U} \to \infty$, $\tilde{t} \to 0$ such that $\tilde{U}\tilde{t} \to 1$. When we write $\Psi = \Phi(r, t)\sin^2\theta$, it follows that

$$\left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2} - \frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right)\Phi = 0\,, \tag{3.3}$$

together with $\Phi = \Phi_r = 0$ on $r = 1$, plus $\Phi \cong \frac{1}{2}r^2\delta(t)$ as $r \to \infty$. It is sufficient to take Laplace transforms and solve the resulting ordinary differential equation to show

$$\Phi = -\frac{3}{2r}\,\mathrm{erf}\left(\frac{r-1}{2t^{1/2}}\right) - \frac{3}{2(\pi t)^{1/2}}\left\{\frac{1}{r} - \exp\left(-\frac{(r-1)^2}{4t}\right)\right\}; \tag{3.4}$$

alternatively, the solution for the impulsively started Stokes flow past a sphere (cf. Landau and Lifschitz [11]) can be differentiated with respect to time. For large times, and $r = O(1)$,

$$\Psi \cong -\tfrac{1}{4}\pi^{-1/2}t^{-3/2}(r^2 - \tfrac{3}{2}r + \tfrac{1}{2}r^{-1})\sin^2\theta\,,$$

which represents the classical Stokes stream function, with a decay rate $O(t^{-3/2})$. Also, it is checked that when we take $r \gg 1$ in (3.4) then the sphere corresponds to a stokeslet of strength $\frac{1}{6}\pi^{-1}$ at large distances.

(a) We now consider the effect of a container on the flow due to a stokeslet pulse. First, the stokeslet is present at the origin inside a circular cylinder, and directed along the z-axis, so that we must solve (3.1) subject to $\Psi = \Psi_\rho = 0$ on $\rho = 1$. When the transforms

$$\bar{\bar{\Psi}}(\rho, \alpha, s) = \int_0^\infty e^{-st}\,dt \int_0^\infty \cos(\alpha z)\Psi(\rho, z, t)\,dz$$

are taken, this leads to

$$4\pi\bar{\bar{\Psi}} = \rho s^{-1}[\alpha K_1(\alpha\rho) - (\alpha^2+s)^{1/2}K_1((\alpha^2+s)^{1/2}\rho) + A(\alpha,s)I_1(\alpha\rho) + B(\alpha,s)I_1((\alpha^2+s)^{1/2}\rho)], \tag{3.5}$$

where the first two terms represent (3.2), and the coefficients A and B are derived from the no-slip conditions on $\rho = 1$ in the form

$$\begin{aligned} &AI_1(\alpha) + BI_1((\alpha^2+s)^{1/2}) = -\alpha K_1(\alpha) + (\alpha^2+s)^{1/2}K_1((\alpha^2+s)^{1/2}), \\ &\alpha AI_0(\alpha) + (\alpha^2+s)^{1/2}BI_0((\alpha^2+s)^{1/2}) = \alpha^2K_0(\alpha) - (\alpha^2+s)^{1/2}K_0((\alpha^2+s)^{1/2}). \end{aligned} \tag{3.6}$$

The general solution is obviously detailed, though it can be confirmed that the inverse Laplace transform of (3.5), given (3.6), is determined solely by the sum of residues at the simple poles, whose positions are known from the roots in the s-plane of the discriminant $\Delta = 0$, where

$$\Delta = (\alpha^2+s)^{1/2}I_1(\alpha)I_0((\alpha^2+s)^{1/2}) - \alpha I_0(\alpha)I_1((\alpha^2+s)^{1/2});$$

these roots all lie on the negative real axis in the s-plane.

When we first limit our discussion to the situation at large distances z, it is seen that

$$\Psi \cong \frac{1}{2\pi^{s/2}t^{1/2}}\sum_{n=1}^{\infty}\frac{\mu_n\{1-J_0(\mu_n)\}}{J_1(\mu_n)}\,\frac{J_1(\mu_n)\rho^2 - \rho J_1(\mu_n\rho)}{J_0(\mu_n) + (\mu_n + 2\mu_n^{-1})J_1(\mu_n)}\,e^{-\mu_n^2 t}, \tag{3.7}$$

where μ_n ($n = 1, 2, 3, \ldots$) are the roots of the equation $\mu J_0(\mu) = 2J_1(\mu)$; numerically, $\mu_1 \cong 5.135$, $\mu_2 \cong 8.406$, $\mu_3 \cong 11.610$. The slowest decay as determined from $n = 1$ shows the axial velocity

$$W \propto \{\mu_1 J_0(\mu_1\rho) - 2J_1(\mu_1)\}\,e^{-\mu_1^2 t} \tag{3.8}$$

from (3.7); the velocity profile is shown in Fig. 5. It is interesting to note that there is this definite flow (with zero mass flux) as $|z| \to \infty$; in the steady Stokes flow with a stokeslet on the axis the velocity decays exponentially in $|z|$ with an infinite sequence of eddies along the length of the cylinder (see Liron and Shahar [12]). Clearly the function of ρ in the axial velocity representation (3.8) follows from the solution of

$$\left(\frac{\partial^2}{\partial\rho^2} - \rho^{-1}\frac{\partial}{\partial\rho} + \mu^2\right)\left(\frac{\partial^2}{\partial\rho^2} - \rho^{-1}\frac{\partial}{\partial\rho}\right)\Phi(\rho) = 0$$

subject to $\Phi \propto \rho$ as $\rho \to 0$ and $\Phi = \Phi' = 0$ on $\rho = 1$ corresponding to the smallest eigenvalue μ_1.

We also observe that when we take the inverse Laplace transform of (3.5) for all z, expressing this as the sum of residues with the slowest decay proportional to $e^{-\mu_1^2 t}$, then the value of μ_1 depends on α within the inverse Fourier transform. It can be shown that $\mu_1(\alpha)$ is a monotonic decreasing function from $\mu_1(0) \cong 5.135$ to $\mu_1(\infty) \cong 3.832$; this fact indicates that the decay is most rapid for large z.

Further, when we consider the effect where the stokeslet, still directed along the z-axis, is positioned at $(c, 0, 0)$ (with $c < 1$) – then it is found that the final motion is still proportional

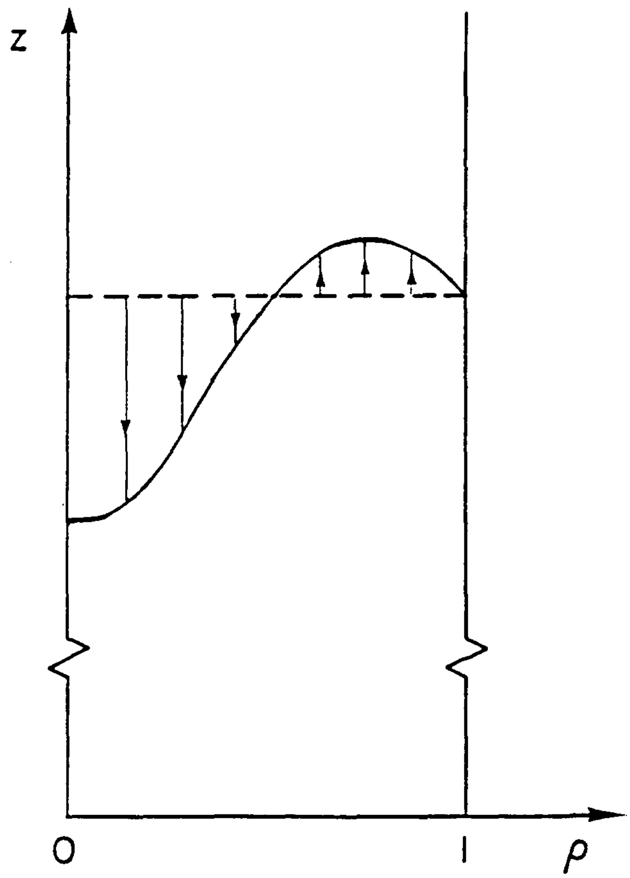

Fig. 5. Velocity profile at large distances for the slowest decaying behaviour for the pulse flow due to a stokeslet on the axis of a circular cylinder.

to (3.8); the axisymmetric behaviour comes to dominate as t increases, as was found for the rotlet in the previous section.

(b) If the stokeslet is at the centre of the sphere $r = 1$, then we must solve (3.3), and the Laplace transform of $\phi(r, t)$, which behaves like (3.2) as $r \to 0$, is given by

$$\begin{aligned} 4\pi\bar{\phi}(r, s) &= \frac{1}{\sigma^2 r} - \left(\frac{1}{\sigma} + \frac{1}{\sigma^2 r}\right) e^{-\sigma r} - \frac{1 - \cosh\sigma + \sigma\sinh\sigma}{\sigma^2} r^2 \\ &\quad - \frac{3 - 3e^{-\sigma} - 3\sigma e^{-\sigma} - \sigma^2 e^{-\sigma}}{\sigma(3\sinh\sigma - 3\sigma\cosh\sigma + \sigma^2\sinh\sigma)} \left(\cosh\sigma r - \frac{\sinh\sigma r}{\sigma r}\right) \\ &\quad - \frac{(\sigma\cosh\sigma - \sinh\sigma)(3 - 3\cosh\sigma + 3\sigma\sinh\sigma - \sigma^2\cosh\sigma)}{\sigma^2(3\sinh\sigma - 3\sigma\cosh\sigma + \sigma^2\sinh\sigma)} r^2 . \end{aligned}$$

The inverse transform is completely given by the residues, and the function $\phi(r, t)$ is seen to be a sum of the terms

$$\left\{\left(\cos\kappa_n r - \frac{\sin\kappa_n r}{\kappa_n r}\right) - \left(\cos\kappa_n - \frac{\sin\kappa_n}{\kappa_n}\right) r^2\right\} e^{-\kappa_n^2 t},$$

where κ_n $(n = 1, 2, 3, \ldots)$ are the roots of $(\kappa^{-1} - \frac{1}{3}\kappa)\tan\kappa = 1$; numerically, $\kappa_1 \cong 5.763$,

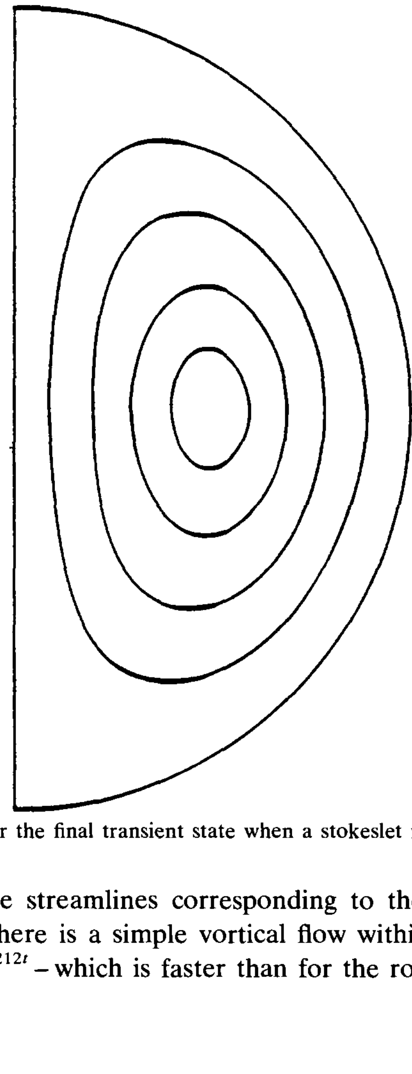

Fig. 6. The streamlines for the final transient state when a stokeslet is at the centre of a sphere.

$\kappa_2 \cong 9.095$, $\kappa_3 \cong 12.323$. The streamlines corresponding to the final transient state where $n = 1$ are given in Fig. 6; there is a simple vortical flow within the sphere, with the rapid decay proportional to $e^{-33.212t}$ – which is faster than for the rotlet inside a sphere.

4. Source flows

The other main model we investigate has a point source at the origin set in the wall along the plane $z = 0$. If the fluid occupies the half-space $z \geqslant 0$, the problem can be posed in terms of a stream function $\Psi(\rho, z, t)$, which satisfies $\Omega_t = L_{-1}\Omega$, $\Omega = L_{-1}\Psi$ subject to $\Psi = \delta(t)$, $\Psi_z = 0$ on $z = 0$, $\Psi \rightarrow 0$ as $z \rightarrow \infty$. When Laplace and Hankel transforms are taken it follows that

$$\Psi = \frac{\rho}{2\pi i} \int_0^\infty kJ_1(k\rho)\,\mathrm{d}k \int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{e}^{st} \left[\left\{ \frac{1}{k} + \frac{1}{(k^2+s)^{1/2} - k} \right\} \mathrm{e}^{-kz} - \frac{\mathrm{e}^{-(k^2+s)^{1/2}z}}{(k^2+s)^{1/2} - k} \right] \mathrm{d}s\,. \tag{4.1}$$

For this case the integrals for the vorticity can be evaluated exactly as

$$\Omega = \frac{\rho^2 \exp\left(-\frac{r^2}{4t}\right)}{16t^3} \left[\left(1 - \frac{z^2}{2t}\right) \exp\left(\frac{\rho^2}{8t}\right) \left\{ I_0\left(\frac{\rho^2}{8t}\right) - I_1\left(\frac{\rho^2}{8t}\right) \right\} - \frac{2z}{(\pi t)^{1/2}} \right].$$

In particular, the integral (4.1) shows that $\Psi \cong (\pi t)^{-1/2} r^{-1} \sin^2\theta$ for $0 < t \ll 1$, which represents a flux from the shear layer which forms instantaneously along the wall towards the origin. The final motion is represented by

$$\Psi \cong \tfrac{1}{32}\rho^2 z^2 t^{-3} \quad \text{as } t \to \infty\,; \tag{4.2}$$

this represents an axisymmetric stagnation point flow towards the origin.

(a) This model is now extended by including a parallel plate along $z = 1$, and so the conditions $\Psi = \Psi_z = 0$ on $z = 1$ must be included. The solution follows previous analyses; the first observation we make concerns the stream function at large distances from the source. We find that Ψ is independent of ρ, and so it represents a radial shear flow where the axial velocity W is zero, and the radial velocity U is proportional to ρ^{-1} with no mass flux. Specifically, for large times t we have

$$\Psi \propto (\sin\delta_1 z \cos\delta_1 z - \delta_1 z + \delta_1 \sin^2\delta_1 z)\, e^{-\delta_1^2 t}\,, \tag{4.3}$$

where $\delta_1 \cong 4.493$ is the smallest root of $\tan\delta = \delta$ (see after (2.10)); the velocity profile corresponding to (4.3) is given in Fig. 7. The representation (4.3) for the stream function can be seen as the solution of $\Psi_{zzzz} + \delta^2\Psi_{zz} = 0$ with $\Psi = \Psi_z = 0$ on $z = 0, 1$ corresponding to the lowest eigenvalue.

The other comment concerns the rates of decay at different positions, parallel to that in Section 3(a). As was described there, when the inverse Laplace transform is taken first for all ρ, the residue corresponding to the slowest decay is proportional to $e^{-\delta_1^2 t}$ where δ_1 depends on the variable k within the Hankel transform. It is found that $\delta_1(k)$ is a monotonic function of k from $\delta_1(0) \cong 4.493$ to $\delta_1(\infty) \cong 3.142$, which shows again that the decay is most rapid at large distances from the source.

(b) To conclude, we remark that the axisymmetric problem with a stokeslet pulse at $(0, c)$ between the walls $z = 0, 1$ has also been investigated, but because the analysis is highly detailed we just comment on the major conclusion that there is no shear flow equivalent to

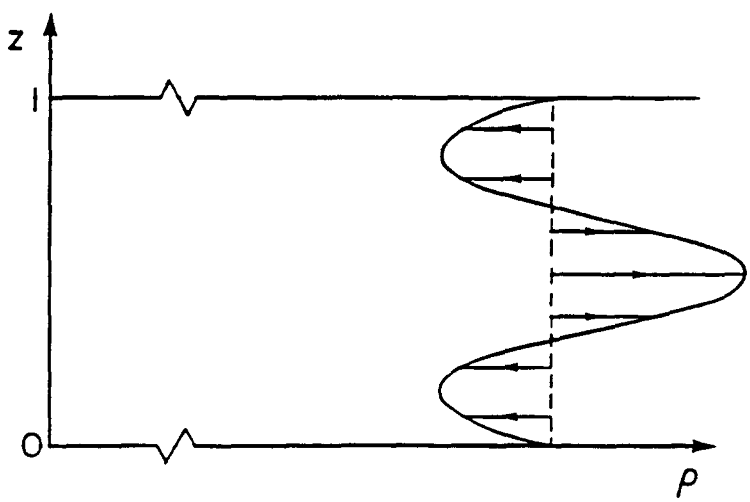

Fig. 7. Velocity profile at large distances for the slowest decaying behaviour for the pulse flow due to a source between parallel plates.

(4.3) – in fact, Ψ decays exponentially as $\rho \to \infty$. To solve the basic equations $\Omega_t = L_{-1}\Omega$, $\Omega = L_{-1}\Psi$ we can write generally

$$\Psi \propto e^{-\delta^2 t} \rho K_1(\mu\rho) H(z) \quad \text{for } \mu > 0\,, \tag{4.4}$$

where $H(z)$ satisfies the Sturm–Liouville problem $\{\partial^2/\partial z^2 + (\delta^2 + \mu^2)\}\{\partial^2/\partial z^2 + \mu^2\}H = 0$, plus $H = H' = 0$ on $z = 0, 1$. Clearly $H(z)$ is the sum of four eigenfunctions $\cos \mu z$, $\sin \mu z$, $\cos \bar{\omega} z$, $\sin \bar{\omega} z$, with $\bar{\omega}^2 = \delta^2 + \mu^2$, where satisfying the boundary conditions leads to the equations for the eigenvalues in the form

$$2\mu\bar{\omega}(1 - \cos \mu \cos \bar{\omega}) - (\mu^2 + \bar{\omega}^2) \sin \mu \sin \bar{\omega} = 0\,. \tag{4.5}$$

This equation (4.5) can be factorized to show either

$$\mu \tan \tfrac{1}{2}\bar{\omega} = \bar{\omega} \tan \tfrac{1}{2}\mu \quad \text{or} \quad \bar{\omega} \tan \tfrac{1}{2}\bar{\omega} = \mu \tan \tfrac{1}{2}\mu\,;$$

each of these does have an infinite set of real, positive roots to provide an infinite set of terms for an eigenfunction expansion. When t is large we would take the solution of the form (4.4) corresponding to the lowest value of δ.

5. Conclusions

Some general observations can be stated now which have emerged from the specific result which have been described so far. First, when the fluid is bounded the decay is exponential in t, whereas it is algebraic otherwise. If the decay is algebraic then the velocity for large t would, of necessity, be of the form $\mathcal{V}(x, y, z)\mathcal{T}(t)$, where $\mathcal{T}(t)$ is algebraically small, and $\mathcal{V}$ is a solution of the steady Stokes flow equation for slow viscous flow with zero velocity on the boundary. From the theorem of Finn and Noll [13], this requires $\mathcal{V}$ to be identically zero within a bounded domain. For an unbounded domain, where it is not essential that $\mathcal{V}$ tends to zero at large distances, algebraic decay is possible.

Therefore, when the fluid is bounded the final motion can be derived from the solution of an eigenvalue problem, where the lowest eigenvalue λ_1 gives the exponent for the decay rate; also, the eigenfunction $\phi_1(\mathbf{r})$ corresponding to λ_1 has fewest nodes of all such eigenfunctions. For second order boundary value problems ϕ_1 has zero nodes (cf. Courant and Hilbert [14]), and because the higher order equations here can be factorized into second order equations the result can be extended. The eigenfunction $\phi_1(\mathbf{r})$ is the 'simplest' one in the sense that it has the greatest degree of symmetry. As an example, when we wanted to find the final motion due to a rotlet or stokeslet positioned generally in a circular cylinder, it was found in each case to be equivalent (up to a multiplicative constant) to that when the rotlet or stokeslet was positioned on the axis. This would also be true for the sphere, and for any container with an axial symmetry; the easiest mathematical problem to solve is also the most relevant to describe the final physical behaviour required.

Because the decay is exponential when the container is bounded, the nature of the final motion can be calculated at least numerically for more complex geometries than would be possible if the solution of the complete initial value problem had to be calculated. If a rotlet is positioned inside a cylinder, for example, both with axes parallel to the z-axis, then it would be sufficient to solve the Strum–Liouville problem $\omega_{xx} + \omega_{yy} + \lambda^2\omega = 0$, where $\omega =$

$\psi_{xx} + \psi_{yy}$, in the domain bounded by the closed curve C which forms the boundary of the cross-section, subject to regularity conditions inside C plus $\psi = \psi_n = 0$ on C.

As the outer boundary of a constrained motion increases in size the exponent in the exponential decay tends to zero and the decay becomes algebraic. In the limiting situation with small exponent the problem has singular perturbation characteristics with two separate domains in the fluid.

For an unbounded domain the final behaviour showed solid body rotation from a rotlet, a uniform stream from a stokeslet, a simple shear flow (Section 2(e)) and stagnation point flow (see (4.2)) along a plane wall. In all cases this represents the 'simplest' flow possible within the geometric constraints corresponding to the pulse; because the decay is algebraic, and the basic variable is the diffusion variable r^2/t (or ρ^2/t, y^2/t, etc., depending on the situation), it is the term with the smallest power in the distance which dominates for large t.

When a pulse acts locally at some point in a partly-bounded container (between parallel plates or within an infinite cylinder), it was found that the rate of decay is greatest at large distances from the pulse.

All the computations described here have been developed within the context of the slow flow assumption so that the Stokes flow equations could describe the motion for all time $t > 0$. However, there is a greater degree of applicability for a large range of situations where the fluid comes to rest freely in the sense that the driving forces stop instantaneously, or else decay rapidly enough for the viscous forces within the fluid to bring the velocity to zero, rather than the rate of decay of these forces to directly control the process. In particular, the flow represented by the eigenfunction corresponding to the smallest eigenvalue in the computations within a bounded container, and the fact that this decays exponentially ensures that such a behaviour must satisfy the linear equations. Alternatively, the unbounded stagnation point flow is intrinsically non-linear throughout the fluid (cf. Proudman and Johnson [15]), and when a large flow for such a geometry is considered the corresponding linear solution would be insufficient; in the neighbourhood of the stagnation point there would be correct local flow at large time, though with the possibility of different algebraic decay.

Finally, a more specific result can be mentioned with respect to pulse flows due to a rotlet set within a cylindrical container; the axis of the rotlet and cylinder are both taken to be parallel to the z-axis and the rotlet is set in the plane $z = 0$. When the transform $\bar{\bar{\psi}}(x, y, \gamma, s)$ for the stream function $\psi(x, y, z, t)$ is defined by $\bar{\bar{\psi}} = \int_0^\infty e^{-st}\, dt \int_0^\infty \psi \cos \gamma z\, dz$, it is seen that $\bar{\bar{\psi}}$ is always a function of x, y and $\gamma^2 + s$ alone. Consequently, when the two-dimensional problem in the x, y-plane for a line rotlet pulse is solved, the corresponding solution for the three-dimensional problem for a point rotlet is just the two-dimensional solution multiplied by $\frac{1}{2}\pi^{1/2}t^{-1/2} \exp(-z^2/4t)$, the Fourier cosine transform of $e^{-\gamma^2 t}$. Therefore, each of the two-dimensional solutions for the impulsively started flow due to a line rotlet described in Smith [9] can be differentiated with respect to time, and then multiplied by this factor to give immediately new solutions for pulse flows in three dimensions. In particular, it follows that there is separated flow for the rotlet both inside and outside a circular cylinder, generalizing the discussion in Section 2(d).

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